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Why nonlocal recursion operators produce local symmetries: new results and applications

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Abstract

It is well known that integrable hierarchies in (1+1) dimensions are local while the recursion operators that generate these hierarchies usually contain nonlocal terms. We resolve this apparent discrepancy by providing simple and universal sufficient conditions for a (nonlocal) recursion operator in (1+1) dimensions to generate a hierarchy of local symmetries. These conditions are satisfied by virtually all recursion operators known today and are much easier to verify than those found in earlier work. We also give explicit formulae for the nonlocal parts of higher recursion, Hamiltonian and symplectic operators of integrable systems in (1+1) dimensions. Using these two results we prove, under some natural assumptions, the Maltsev–Novikov conjecture stating that higher Hamiltonian, symplectic and recursion operators of integrable systems in (1+1) dimensions are weakly nonlocal, i.e., the coefficients of these operators are local and these operators contain at most one integration operator in each term.

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Introduction

It is common knowledge that an integrable system of PDEs never comes alone—it always is a member of an infinite integrable hierarchy. In particular, if we deal with evolution systems then the members of the hierarchy are symmetries for each other, and using a recursion operator, which maps symmetries to symmetries, offers a natural way to construct the whole infinite hierarchy from a single seed system, see e.g. [1–3] and references therein and cf [2–7] and references therein for the hierarchies generated by master symmetries.

The overwhelming majority of recursion operators in (1+1) dimensions share two key features [1–3, 8]: they are *hereditary*, i.e., their Nijenhuis torsion vanishes [9], and *weakly nonlocal* [10], i.e., all their nonlocal terms have the form $a \otimes D^{-1} \circ b$, where a and b are local functions, possibly vector valued, and D is the operator of the total x-derivative; see below for details.

On the other hand, it is well known that nearly all integrable hierarchies in (1+1) dimensions are *local*. Usually it is not difficult to check that applying the recursion operator to a local seed symmetry once or twice yields local quantities, but the locality of the whole *infinite* hierarchy is quite difficult to verify rigorously.

It is therefore natural to ask [11] whether a weakly nonlocal hereditary operator will always produce a local hierarchy, as in earlier work [3, 11–16] one always had to require the existence of some nontrivial additional structures (e.g., the scaling symmetry [11, 15, 16] or bi-Hamiltonian structure [3, 12]) in order to get the proof of locality through. We show that this is not necessary: theorem 1 states that if for a normal weakly nonlocal hereditary recursion operator $\mathfrak{R}$ the Lie derivative $L_{\boldsymbol{Q}}(\mathfrak{R})$ of $\mathfrak{R}$ along a local symmetry $\boldsymbol{Q}$ vanishes[1] and $\mathfrak{R}(\boldsymbol{Q})$ is local, then $\mathfrak{R}^j(\boldsymbol{Q})$ are local for all $j = 2, 3, \ldots$. Note that, unlike e.g. [11, 16], we do *not* require the hierarchy in question to be time independent, and our proposition 1 and theorem 1 can be successfully employed for proving locality of the so-called variable coefficients hierarchies, including for instance those constructed in [17, 18] and [2], cf example 2.

Given an operator $\mathfrak{R}$, it is usually immediate whether it is weakly nonlocal, but it can be quite difficult to check whether it is hereditary, especially if we deal with newly discovered integrable systems with no multi-Hamiltonian representation and no Lax pair known. Amazingly enough, the existence of a scaling symmetry shared by $\mathfrak{R}$ and $\boldsymbol{Q}$ enables us to avoid the cumbersome direct verification of whether $\mathfrak{R}$ is hereditary and allows us to prove locality *and* commutativity of the corresponding hierarchy in a very simple and straightforward manner, as shown in proposition 3 and corollary 3. This is in a sense reminiscent of the construction of compatible Hamiltonian operators via infinitesimal deformations in Smirnov [19] (see also [20] and references therein) and is quite different from the approach of [11], where both $\mathfrak{R}$ being hereditary and existence of scaling symmetry were required *ab initio*.

Let $\mathfrak{R}$, $\mathfrak{P}$ and $\mathfrak{S}$ be respectively recursion, Hamiltonian and symplectic operator for some (1+1)-dimensional integrable system, and let all of them be weakly nonlocal. Motivated by the examples of nonlinear Schrödinger and KdV equations, Maltsev and Novikov [10] conjectured that higher recursion operators $\mathfrak{R}^k$, higher Hamiltonian operators $\mathfrak{P} \circ \mathfrak{R}^{\dagger k}$ and higher symplectic operators $\mathfrak{S} \circ \mathfrak{R}^k$ are weakly nonlocal for all $k \in \mathbb{N}$ as well.

Combining our corollary 2 with the results of Enriquez, Orlov and Rubtsov [21] enabled us to prove this conjecture under some natural assumptions, the most important of which is that $\mathfrak{R}$ is hereditary, see theorem 2 for details. This has interesting and quite far-reaching consequences for both theory and applications of integrable systems, e.g., in connection with the so-called Whitham averaging, cf discussion in [10, 22, 23].

1. Preliminaries

Denote by $\mathcal{A}_j$ the algebra of locally analytic functions of $x, t, \boldsymbol{u}, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_j$ under the standard multiplication, and let $\mathcal{A} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. We shall refer to the elements of $\mathcal{A}$ as *local* functions

[1] Where does the condition $L_{\boldsymbol{Q}}(\mathfrak{R}) = 0$ come from? As all members of an integrable hierarchy must be compatible, the symmetries $\mathfrak{R}^i(\boldsymbol{Q})$ must commute, and this is ensured by requiring that $\mathfrak{R}$ be hereditary and that $L_{\boldsymbol{Q}}(\mathfrak{R}) = 0$, cf e.g. [9]. Moreover, $L_{\boldsymbol{Q}}(\mathfrak{R}) = 0$ means that $\mathfrak{R}$ is a recursion operator for the evolution system $\boldsymbol{u}_\tau = \boldsymbol{Q}$.

[24–27]. Here $\boldsymbol{u}_k = \left(u_k^1, \ldots, u_k^s\right)^T$ are s-component vectors, $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, and the superscript T stands for the matrix transposition. The derivation [1, 25]

$$D \equiv D_x = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} \boldsymbol{u}_{j+1} \cdot \frac{\partial}{\partial \boldsymbol{u}_j}$$

makes $\mathcal{A}$ into a differential algebra. Informally, x plays the role of the space variable, and D is the total x-derivative, cf e.g. [1, 25]. It is closely related to the operator of variational derivative [1–3]

$$\frac{\delta}{\delta \boldsymbol{u}} = \sum_{j=0}^{\infty} (-D)^j \frac{\partial}{\partial \boldsymbol{u}_j}.$$

In particular, see e.g. [1, 3], for any $f \in \mathcal{A}$ we have

$$\frac{\delta f}{\delta \boldsymbol{u}} = 0 \qquad \text{if and only if} \quad f \in \operatorname{Im} D. \tag{1}$$

Here and below '$\cdot$' stands for the scalar product of two s-component vectors, and $\operatorname{Im} D$ denotes the image of D in $\mathcal{A}$, so $f \in \operatorname{Im} D$ means that $f = D(g)$ for some $g \in \mathcal{A}$.

For a (scalar, vector or matrix) local function f define [1] its *order* $\operatorname{ord} f$ as the greatest integer k such that $\partial f / \partial \boldsymbol{u}_k \neq 0$ (if $f = f(x,t)$, we set $\operatorname{ord} f = 0$ by definition), and define the *directional derivative* of f (cf e.g. [1, 9]) by the formula

$$f' = \sum_{i=0}^{\infty} \frac{\partial f}{\partial \boldsymbol{u}_i} D^i.$$

Consider now the algebra $\operatorname{Mat}_q(\mathcal{A})[\![D^{-1}]\!]$ of formal series of the form $\mathfrak{H} = \sum_{j=-\infty}^{k} h_j D^j$, where h_j are $q \times q$ matrices with entries from $\mathcal{A}$. The multiplication law in this algebra is given by the (extended by linearity) the generalized Leibniz rule [1, 24, 26, 27]:

$$a D^i \circ b D^j = a \sum_{q=0}^{\infty} \frac{i(i-1)\cdots(i-q+1)}{q!} D^q(b) D^{i+j-q}. \tag{2}$$

The commutator $[\mathfrak{A}, \mathfrak{B}] = \mathfrak{A} \circ \mathfrak{B} - \mathfrak{B} \circ \mathfrak{A}$ further makes $\operatorname{Mat}_q(\mathcal{A})[\![D^{-1}]\!]$ into a Lie algebra.

Recall [1, 24, 26, 27] that the *degree* $\deg \mathfrak{H}$ of $\mathfrak{H} = \sum_{j=-\infty}^{p} h_j D^j \in \operatorname{Mat}_q(\mathcal{A})[\![D^{-1}]\!]$ is the greatest integer m such that $h_m \neq 0$. For any $\mathfrak{H} = \sum_{j=-\infty}^{m} h_j D^j \in \operatorname{Mat}_q(\mathcal{A})[\![D^{-1}]\!]$ define its differential part $\mathfrak{H}_+ = \sum_{j=0}^{m} h_j D^j$ and nonlocal part $\mathfrak{H}_- = \sum_{j=-\infty}^{-1} h_j D^j$ so that $\mathfrak{H}_- + \mathfrak{H}_+ = \mathfrak{H}$, and let $\mathfrak{H}^\dagger = \sum_{j=-\infty}^{m} (-D)^j \circ h_j^T$ stand for the formal adjoint of $\mathfrak{H}$, see e.g. [1, 24, 26, 27].

We shall employ the notation $\mathcal{A}^q$ for the space of q-component functions with entries from $\mathcal{A}$, no matter whether they are interpreted as column or row vectors. Following [10], we shall call $\mathfrak{H} \in \operatorname{Mat}_q(\mathcal{A})[\![D^{-1}]\!]$ *weakly nonlocal* if there exist $\vec{f}_\alpha \in \mathcal{A}^q$, $\vec{g}_\alpha \in \mathcal{A}^q$ and $k \in \mathbb{N}$ such that $\mathfrak{H}_-$ can be written in the form $\mathfrak{H}_- = \sum_{\alpha=1}^{k} \vec{f}_\alpha \otimes D^{-1} \circ \vec{g}_\alpha$. We shall further say that $\mathfrak{H} \in \operatorname{Mat}_q(\mathcal{A})[\![D^{-1}]\!]$ is *local* (or *purely differential*) if $\mathfrak{H}_- = 0$. Nearly all recursion operators known today in (1+1) dimensions, as well as Hamiltonian and symplectic operators, are weakly nonlocal, cf e.g. [8].

The space $\mathcal{V}$ of s-component columns with entries from $\mathcal{A}$ is made into a Lie algebra if we set $[\boldsymbol{P}, \boldsymbol{Q}] = \boldsymbol{Q}'(\boldsymbol{P}) - \boldsymbol{P}'(\boldsymbol{Q})$, see e.g. [1, 2, 9, 24]. The Lie derivative of $\boldsymbol{R} \in \mathcal{V}$ along $\boldsymbol{Q} \in \mathcal{V}$ is then given [1, 2, 3, 28] by $L_{\boldsymbol{Q}}(\boldsymbol{R}) = [\boldsymbol{Q}, \boldsymbol{R}]$. The natural dual of $\mathcal{V}$ is the space $\mathcal{V}^*$ of s-component rows with entries from $\mathcal{A}$. For $\gamma \in \mathcal{V}^*$ we define [2, 3, 11, 28] its Lie derivative along $\boldsymbol{Q} \in \mathcal{V}$ as $L_{\boldsymbol{Q}}(\gamma) = \gamma'(\boldsymbol{Q}) + \boldsymbol{Q}'^{\dagger}(\gamma)$, see [3, 28] for more details and for the related complex of formal calculus of variations.

For $Q \in \mathcal{V}$ and $\gamma \in \mathcal{V}^*$ we have, see e.g. [1], $\delta(Q \cdot \gamma)/\delta u = Q'^{\dagger}(\gamma) + \gamma'^{\dagger}(Q)$; hence if $\gamma'^{\dagger}(Q) = \gamma'(Q)$ then

$$L_Q(\gamma) = \delta(Q \cdot \gamma)/\delta u. \tag{3}$$

If $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$, $\mathfrak{S} : \mathcal{V} \to \mathcal{V}^*$, $\mathfrak{P} : \mathcal{V}^* \to \mathcal{V}$, $\mathfrak{N} : \mathcal{V}^* \to \mathcal{V}^*$ are weakly nonlocal or, even more broadly, belong to $\mathrm{Mat}_s(\mathcal{A})[\![D^{-1}]\!]$, then we can [2, 5, 9] define their Lie derivatives along $Q \in \mathcal{V}$ as follows: $L_Q(\mathfrak{R}) = \mathfrak{R}'[Q] - [Q', \mathfrak{R}]$, $L_Q(\mathfrak{N}) = \mathfrak{N}'[Q] + [Q'^{\dagger}, \mathfrak{N}]$, $L_Q(\mathfrak{P}) = \mathfrak{P}'[Q] - Q' \circ \mathfrak{P} - \mathfrak{P} \circ Q'^{\dagger}$, $L_Q(\mathfrak{S}) = \mathfrak{S}'[Q] + Q'^{\dagger} \circ \mathfrak{S} + \mathfrak{S} \circ Q'$, where for $\mathfrak{H} = \sum_{j=-\infty}^{m} h_j D^j$ we set $\mathfrak{H}'[Q] = \sum_{j=-\infty}^{m} h'_j[Q] D^j$. Here and below we do *not* assume $\mathfrak{R}$ and $\mathfrak{S}$ (resp. $\mathfrak{P}$ and $\mathfrak{N}$) to be necessarily defined on the whole of $\mathcal{V}$ (resp. on the whole of $\mathcal{V}^*$).

An operator $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ is called *hereditary* [9] (or *Nijenhuis* [3]) on a linear subspace $\mathcal{L}$ of the domain of definition of $\mathfrak{R}$ if for all $Q \in \mathcal{L}$

$$L_{\mathfrak{R}(Q)}(\mathfrak{R}) = \mathfrak{R} \circ L_Q(\mathfrak{R}). \tag{4}$$

In what follows, by saying that $\mathfrak{R}$ is hereditary without specifying $\mathcal{L}$ we shall mean that $\mathfrak{R}$ is hereditary on its whole domain of definition, cf e.g. [9]. If $\mathfrak{R}$ is hereditary on $\mathcal{L}$, then for any $Q \in \mathcal{L}$ such that $\mathfrak{R}^k(Q) \in \mathcal{L}$ for all $k \in \mathbb{N}$ we have $[\mathfrak{R}^i(Q), \mathfrak{R}^j(Q)] = 0$, $i, j = 0, 1, 2, \ldots$, cf e.g. [2, 5]. We do not address here the issue of proper definition of $\mathfrak{R}^j(Q)$ and refer the reader to [29–31] and [32] and references therein for details.

Denote by $\mathcal{S}(\mathfrak{R}, Q)$ the linear span of $\mathfrak{R}^i(Q)$, $i = 0, 1, 2, \ldots$. We readily see from (4) that $L_{\mathfrak{R}^i(Q)}(\mathfrak{R}) = 0$ for all $i = 0, 1, 2, \ldots$ if and only if $L_Q(\mathfrak{R}) = 0$ and $\mathfrak{R}$ is hereditary on $\mathcal{S}(\mathfrak{R}, Q)$. Hence, if $L_{\mathfrak{R}^i(Q)}(\mathfrak{R}) = 0$ for all $i = 0, 1, 2, \ldots$, then $[\mathfrak{R}^i(Q), \mathfrak{R}^j(Q)] = 0$ for all $i, j = 0, 1, 2, \ldots$.

2. The main result and its applications

Consider a weakly nonlocal operator $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ of the form

$$\mathfrak{R} = \sum_{i=0}^{r} a_i D^i + \sum_{\alpha=1}^{p} G_\alpha \otimes D^{-1} \circ \gamma_\alpha, \tag{5}$$

where a_i are $s \times s$ matrices with entries from $\mathcal{A}$, $G_\alpha \in \mathcal{V}$, $\gamma_\alpha \in \mathcal{V}^*$ and $r \geqslant 0$.

We shall call $\mathfrak{R}$ of the form (5) *normal* if for all $\alpha, \beta = 1, \ldots, p$ we have $\gamma'_\alpha = \gamma'^{\dagger}_\alpha$, $\zeta'_\alpha = \zeta'^{\dagger}_\alpha$, where $\zeta_\alpha = \mathfrak{R}^{\dagger}(\gamma_\alpha)$, and $L_{G_\alpha}(\gamma_\beta) = 0$. This is a very common property: it appears that all known-today weakly nonlocal hereditary recursion operators of integrable systems in (1+1) dimensions are normal.

Proposition 1. *Consider a normal $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ of the form (5), and let $Q \in \mathcal{V}$ and $\mathfrak{R}$ be such that $\mathfrak{R}$ is hereditary on $\mathcal{S}(\mathfrak{R}, Q)$, $L_Q(\mathfrak{R}) = 0$ and $L_Q(\gamma_\alpha) = 0$ for all $\alpha = 1, \ldots, p$.*

Then $Q_j = \mathfrak{R}^j(Q)$ are local and commute for all $j = 0, 1, 2, \ldots$.

Proof. The commutativity of Q_j immediately follows from $\mathfrak{R}$ being hereditary on $\mathcal{S}(\mathfrak{R}, Q)$, see above. Now assume that Q_j is local and $L_{Q_j}(\gamma_\alpha) = 0$, and let us show that Q_{j+1} is local and $L_{Q_{j+1}}(\gamma_\alpha) = 0$. First of all, by (3) we have $\delta(Q_j \cdot \gamma_\alpha)/\delta u = L_{Q_j}(\gamma_\alpha) = 0$, so by (1) $Q_j \cdot \gamma_\alpha \in \operatorname{Im} D$ for all $\alpha = 1, \ldots, p$, and hence $Q_{j+1} = \mathfrak{R}(Q_j)$ is local.

To proceed, we need the following lemma:

Lemma 1. *Let $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ of the form (5) and $Q \in \mathcal{V}$ be such that $L_{G_\alpha}(\gamma_\beta) = 0$, $L_Q(\gamma_\alpha) = 0$ and $\gamma'^{\dagger}_\alpha(Q) = \gamma'_\alpha(Q)$ for all $\alpha, \beta = 1, \ldots, p$.*

Then $L_{\mathfrak{R}(Q)}(\gamma_\alpha) = \delta(Q \cdot \mathfrak{R}^{\dagger}(\gamma_\alpha))/\delta u$ for all $\alpha = 1, \ldots, p$.

Proof of the lemma. As $\gamma_\alpha'^{\dagger}(Q) = \gamma_\alpha'(Q)$, by (3) we have $\delta(Q \cdot \gamma_\alpha)/\delta u = L_Q(\gamma_\alpha) = 0$, so by (1) $Q \cdot \gamma_\alpha = D(f_\alpha)$ for some $f_\alpha \in \mathcal{A}$. Likewise, $G_\beta \cdot \gamma_\alpha = D(g_{\alpha\beta})$ for some $g_{\alpha\beta} \in \mathcal{A}$, whence

$$\mathfrak{R}(Q) \cdot \gamma_\alpha = Q \cdot \mathfrak{R}^\dagger(\gamma_\alpha) + D\left(\sum_{i=1}^{r}\sum_{j=0}^{i-1}(-D)^j\left(a_i^T\gamma_\alpha\right) \cdot D^{i-j-1}(Q) + \sum_{\beta=1}^{p} g_{\alpha\beta} f_\beta\right).$$

Using this formula along with (1) and (3) yields $L_{\mathfrak{R}(Q)}(\gamma_\alpha) = \delta(\mathfrak{R}(Q) \cdot \gamma_\alpha)/\delta u = \delta(Q \cdot \mathfrak{R}^\dagger(\gamma_\alpha))/\delta u$. The lemma is proved.

As $\mathfrak{R}$ is hereditary on $\mathcal{S}(\mathfrak{R}, Q)$, repeatedly using (4) yields $L_{Q_j}(\mathfrak{R}) = L_{\mathfrak{R}^j(Q)}(\mathfrak{R}) = \mathfrak{R}^j \circ L_Q(\mathfrak{R}) = 0$. Next, using lemma 1, the normality of $\mathfrak{R}$, the equality $\zeta_\alpha' = \zeta_\alpha'^{\dagger}$, where $\zeta_\alpha = \mathfrak{R}^\dagger(\gamma_\alpha)$, and (3), we obtain $L_{Q_{j+1}}(\gamma_\alpha) = L_{\mathfrak{R}(Q_j)}(\gamma_\alpha) = L_{Q_j}(\mathfrak{R}^\dagger(\gamma_\alpha)) = L_{Q_j}(\mathfrak{R}^\dagger)\gamma_\alpha + \mathfrak{R}^\dagger L_{Q_j}(\gamma_\alpha) = L_{Q_j}(\mathfrak{R}^\dagger)\gamma_\alpha = \left(L_{Q_j}(\mathfrak{R})\right)^\dagger\gamma_\alpha = 0$. The induction on j starting from $j = 0$ completes the proof. □

If $G_\alpha, \alpha = 1, \ldots, p$, are linearly independent over the field $\mathbb{T}$ of locally analytic functions of t (note that this can always be assumed without loss of generality), then the conditions $L_Q(\gamma_\alpha) = 0, \alpha = 1, \ldots, p$, are equivalent to the requirement that $\mathfrak{R}(Q)$ is local, and we arrive at the result announced in the introduction.

Theorem 1. *Let $G_\alpha, \alpha = 1, \ldots, p$, be linearly independent over the field $\mathbb{T}$ of locally analytic functions of t. Suppose that a normal weakly nonlocal $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ of the form (5) and $Q \in \mathcal{V}$ are such that $L_Q(\mathfrak{R}) = 0$, $\mathfrak{R}$ is hereditary on $\mathcal{S}(\mathfrak{R}, Q)$, and $\mathfrak{R}(Q)$ is local.*

Then the quantities $Q_j = \mathfrak{R}^j(Q)$ are local for all $j = 2, 3, \ldots,$ and $[Q_j, Q_k] = 0$ for all $j, k = 0, 1, 2 \ldots.$

Proof. By virtue of proposition 1 it is enough to show that if $\mathfrak{R}(Q)$ is local then $L_Q(\gamma_\alpha) = 0$ for all $\alpha = 1, \ldots, p$. To prove this, suppose that $\mathfrak{R}(Q)$ is local but for some value(s) of α we have $L_Q(\gamma_\alpha) \neq 0$.

Then we have $\mathfrak{R}(Q) = M + \sum_{\alpha=1}^{p} G_\alpha\omega_\alpha$, where M is local, and ω_α denotes the nonlocal part of $D^{-1}(\gamma_\alpha \cdot Q)$ (some of ω_α may be zeros). By assumption, $\mathfrak{R}(Q)$ is local, so $\sum_{\alpha=1}^{p} G_\alpha\omega_\alpha = 0$. Moreover, as $D^i(\mathfrak{R}(Q)), i = 1, 2, \ldots,$ are local too, we arrive at the following system of algebraic equations for ω_α:

$$\sum_{\alpha=1}^{p} D^j(G_\alpha)\omega_\alpha = 0, \qquad j = 0, 1, 2, \ldots.$$

This system has the same structure as (A.2), and using the linear independence of G_α over $\mathbb{T}$ we conclude, in analogy with the proof of lemma 2 from the appendix, that $\omega_\alpha = 0$ for all $\alpha = 1, \ldots, p$. Hence $\gamma_\alpha \cdot Q \in \operatorname{Im} D$ and by (1) we have $\delta(\gamma_\alpha \cdot Q)/\delta u = 0$. Finally, as $\gamma_\alpha'^{\dagger} = \gamma_\alpha'$ by assumption, (3) yields $L_Q(\gamma_\alpha) = 0$ for all $\alpha = 1, \ldots, p$, as required. □

The seed symmetry Q often commutes with G_α: $L_Q(G_\alpha) \equiv [Q, G_\alpha] = 0$. Then we can bypass the check of the conditions $L_Q(\gamma_\alpha) = 0$ in proposition 1 as follows.

Corollary 1. *If $G_\alpha, \alpha = 1, \ldots, p$, are linearly independent over the field $\mathbb{T}$ of locally analytic functions of t, then for any $\mathfrak{R}$ of the form (5) and any $Q \in \mathcal{V}$ such that $L_Q(\mathfrak{R}) = 0$ and $L_Q(G_\alpha) = 0$ for all $\alpha = 1, \ldots, p$ we have $L_Q(\gamma_\alpha) = 0, \alpha = 1, \ldots, p$.*

Proof. Indeed, $(L_Q(\mathfrak{R}))_- = \sum_{\alpha=1}^{p}(G_\alpha \otimes D^{-1} \circ L_Q(\gamma_\alpha) + L_Q(G_\alpha) \otimes D^{-1} \circ \gamma_\alpha) = \sum_{\alpha=1}^{p} G_\alpha \otimes D^{-1} \circ L_Q(\gamma_\alpha)$. As $L_Q(\mathfrak{R}) = 0$ implies $(L_Q(\mathfrak{R}))_- = 0$, we get $\sum_{\alpha=1}^{p} G_\alpha \otimes D^{-1} \circ L_Q(\gamma_\alpha) = 0$, whence by linear independence of $\boldsymbol{G}_\alpha$ over $\mathbb{T}$ and lemma 2 (see the appendix) we obtain $L_{\boldsymbol{Q}}(\gamma_\alpha) = 0$, as required. □

We also have the following 'dual' of proposition 1 for the elements of $\mathcal{V}^*$.

Proposition 2. *Consider a hereditary operator* $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ *of the form (5) and assume that* $L_{\boldsymbol{G}_\alpha}(\mathfrak{R}) = 0$ *for all* $\alpha = 1, \ldots, p$. *Let* $\boldsymbol{\zeta} \in \mathcal{V}^*$ *be such that* $L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}) = 0$ *for all* $\alpha = 1, \ldots, p$, $\boldsymbol{\zeta}' = \boldsymbol{\zeta}'^\dagger$ *and* $(\mathfrak{R}^\dagger(\boldsymbol{\zeta}))' = (\mathfrak{R}^\dagger(\boldsymbol{\zeta}))'^\dagger$.

Then $\boldsymbol{\zeta}_j = \mathfrak{R}^{\dagger j}(\boldsymbol{\zeta})$ *are local, i.e.,* $\boldsymbol{\zeta}_j \in \mathcal{V}^*$, *and satisfy* $\boldsymbol{\zeta}_j' = \boldsymbol{\zeta}_j'^\dagger$ *for all* $j \in \mathbb{N}$.

Proof. Again, assume that $\boldsymbol{\zeta}_j$ is local, $L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}_j) = 0$ and $\boldsymbol{\zeta}_j' = \boldsymbol{\zeta}_j'^\dagger$, and let us prove that $\boldsymbol{\zeta}_{j+1}$ is local as well, $L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}_{j+1}) = 0$, and $\boldsymbol{\zeta}_{j+1}' = \boldsymbol{\zeta}_{j+1}'^\dagger$.

As $\mathfrak{R}$ is hereditary, the equalities $\boldsymbol{\zeta}' = \boldsymbol{\zeta}'^\dagger$ and $(\mathfrak{R}^\dagger(\boldsymbol{\zeta}))' = (\mathfrak{R}^\dagger(\boldsymbol{\zeta}))'^\dagger$ imply [5, 9] that $\boldsymbol{\zeta}_j' = \boldsymbol{\zeta}_j'^\dagger$ for all $j = 0, 1, 2, \ldots$. We further have $L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}_{j+1}) = L_{\boldsymbol{G}_\alpha}(\mathfrak{R}^\dagger \boldsymbol{\zeta}_j) = L_{\boldsymbol{G}_\alpha}(\mathfrak{R}^\dagger)\boldsymbol{\zeta}_j + \mathfrak{R}^\dagger L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}_j) = L_{\boldsymbol{G}_\alpha}(\mathfrak{R}^\dagger)\boldsymbol{\zeta}_j = (L_{\boldsymbol{G}_\alpha}(\mathfrak{R}))^\dagger \boldsymbol{\zeta}_j = 0$, as desired.

Finally, $\delta(\boldsymbol{\zeta}_j \cdot \boldsymbol{G}_\alpha)/\delta \boldsymbol{u} = L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}_j) = 0$ implies, by virtue of (1), that $\boldsymbol{G}_\alpha \cdot \boldsymbol{\zeta}_j \in \operatorname{Im} D$, and hence $\boldsymbol{\zeta}_{j+1}$ is indeed local. The induction on j completes the proof. □

Corollary 2. *Let an operator* $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ *of the form (5) be hereditary and normal, and let* $L_{\boldsymbol{G}_\alpha}(\mathfrak{R}) = 0, \alpha = 1, \ldots, p$.

Then $\boldsymbol{\zeta}_{\alpha,j} = \mathfrak{R}^{\dagger j}(\gamma_\alpha)$ *and* $\boldsymbol{G}_{\alpha,j} = \mathfrak{R}^j(\boldsymbol{G}_\alpha)$ *are local,* $\boldsymbol{\zeta}_{\alpha,j}' = \boldsymbol{\zeta}_{\alpha,j}'^\dagger$ *and* $[\boldsymbol{G}_{\alpha,j}, \boldsymbol{G}_{\alpha,k}] = 0$ *for all* $j, k = 0, 1, 2, \ldots,$ *and* $\alpha = 1, \ldots, p$.

3. Hereditary operators and scaling

Given an $\boldsymbol{S} \in \mathcal{V}$, if $L_{\boldsymbol{S}}(K) = \kappa K$ for some constant κ, then K is said to be of weight κ (with respect to the scaling $\boldsymbol{S}$), and we write $\kappa = \mathrm{wt}_{\boldsymbol{S}}(K)$, cf e.g. [11].

Proposition 3. *Let* $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ *and* $\boldsymbol{Q} \in \mathcal{V}$ *be such that* $L_{\boldsymbol{Q}}(\mathfrak{R}) = 0$. *Suppose that* $\mathfrak{R}$ *has the form (5),* $r \equiv \deg \mathfrak{R} > 0$, $L_{\boldsymbol{Q}}(\gamma_\alpha) = 0$ *for all* $\alpha = 1, \ldots, p$, $q \equiv \operatorname{ord} \boldsymbol{Q} > \max(\operatorname{ord} a_r - r, 1)$, *the matrix* $\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q$ *has s distinct eigenvalues, and* $\det \partial \boldsymbol{Q}/\partial \boldsymbol{u}_q \neq 0$. *Further assume that there exist a nonzero constant* ζ *and an s-component vector function* $\boldsymbol{S}_0(\boldsymbol{u})$ *such that for* $\boldsymbol{S} = x\boldsymbol{u}_1 + \boldsymbol{S}_0(\boldsymbol{u})$ *we have* $L_{\boldsymbol{S}}(\mathfrak{R}) = r\zeta\mathfrak{R}$, $L_{\boldsymbol{S}}(\boldsymbol{Q}) = q\zeta\boldsymbol{Q}$, *and there exists an* $s \times s$ *matrix* Γ *with entries from* $\mathcal{A}$ *that simultaneously diagonalizes* $\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q$ *and* $\partial \boldsymbol{S}_0/\partial \boldsymbol{u}$ *and satisfies* $\Gamma'[\boldsymbol{S}] - xD(\Gamma) = 0$.

Then $L_{\mathfrak{R}^j(\boldsymbol{Q})}(\mathfrak{R}) = 0$ *for all* $j = 1, 2 \ldots,$ *and hence* $\mathfrak{R}$ *is hereditary on* $\mathcal{S}(\mathfrak{R}, \boldsymbol{Q})$ *and* $[\mathfrak{R}^i(\boldsymbol{Q}), \mathfrak{R}^j(\boldsymbol{Q})] = 0$ *for all* $i, j = 0, 1, 2, \ldots$.

Proof. Consider an algebra $\tilde{\mathcal{A}}$ of all locally analytic functions that depend on x, t, a finite number of $\boldsymbol{u}_j$, and a finite number of nonlocal variables from the universal Abelian covering over the system $\boldsymbol{u}_\tau = \boldsymbol{Q}$, see [32, 33] and references therein for more details on this covering. Let $\mathfrak{L} \equiv \sum_{i=-\infty}^m b_i D^i$, where b_i are $s \times s$ matrices with entries from $\tilde{\mathcal{A}}$, satisfy $\mathfrak{L}'[\boldsymbol{Q}] - [\boldsymbol{Q}', \mathfrak{L}] = 0$.

Assume first that $s = 1$. Then, as $q > 1$, equating to zero the coefficient at D^{m+q-1} in $\mathfrak{L}'[\boldsymbol{Q}] - [\boldsymbol{Q}', \mathfrak{L}] = 0$ yields $q\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q D(b_m) - m b_m D(\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q) = 0$, or equivalently $D(b_m(\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q)^{-m/q}) = 0$. In complete analogy with proposition 5 of [32], the kernel of D in $\tilde{\mathcal{A}}$ is readily seen to be exhausted by the functions of t and τ. Hence $b_m = c_m(t, \tau)(\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q)^{m/q}$ for some function $c_m(t, \tau)$.

For $s > 1$ a similar computation shows that there exists (cf e.g. [24, 26]) a diagonal $s \times s$ matrix $c_m(t, \tau)$ such that $b_m = \Gamma^{-1} c_m(t, \tau)\Lambda^{m/q}\Gamma$, where Γ is a matrix bringing $\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q$ into the diagonal form, i.e., $\Gamma \partial \boldsymbol{Q}/\partial \boldsymbol{u}_q \Gamma^{-1} = \mathrm{diag}(\lambda_1, \ldots, \lambda_s) \equiv \Lambda$, where λ_i are the eigenvalues of $\partial \boldsymbol{Q}/\partial \boldsymbol{u}_q$, and $\Lambda^{m/q} = \mathrm{diag}\left(\lambda_1^{m/q}, \ldots, \lambda_s^{m/q}\right)$.

It is straightforward to verify that $\mathfrak{L}_j \equiv L_{\mathfrak{R}^j(\boldsymbol{Q})}(\mathfrak{R})$, for $j = 1, 2, \ldots,$ satisfy $L_{\boldsymbol{Q}}(\mathfrak{L}_j) \equiv \mathfrak{L}_j'[\boldsymbol{Q}] - [\boldsymbol{Q}', \mathfrak{L}_j] = 0$. Moreover, under the assumptions made $\mathfrak{R}$ is a recursion operator for the system $\boldsymbol{u}_\tau = \boldsymbol{Q}$, and, as $L_{\boldsymbol{Q}}(\gamma_\alpha) = 0$ for all $\alpha = 1, \ldots, p$, by proposition 2 of [32] we have $\mathfrak{R}^j(\boldsymbol{Q}) \in \tilde{\mathcal{A}}^s$ for all $j \in \mathbb{N}$. Then, using the above formulae for the leading coefficients of $\mathfrak{L}_j$ and the condition $\Gamma'[\boldsymbol{S}] - xD(\Gamma) = 0$ along with the assumption that Γ diagonalizes $\partial \boldsymbol{S}_0/\partial \boldsymbol{u}$, we readily find that $\mathrm{wt}_{\boldsymbol{S}}(\mathfrak{L}_j) = \zeta \deg \mathfrak{L}_j$.

As $q > 1$, equating to zero the coefficient at D^{r+q} on the l.h.s. of $L_{\boldsymbol{Q}}(\mathfrak{R}) = 0$, we conclude that the leading coefficient $\Phi \equiv \partial \boldsymbol{Q}/\partial \boldsymbol{u}_q$ of the formal series $\boldsymbol{Q}'$ commutes with the leading coefficient a_r of $\mathfrak{R}$. Moreover, as $q > \mathrm{ord}\, a_r - r$, the same is true for the leading coefficient $a_r^j \Phi$ of $(\mathfrak{R}^j(\boldsymbol{Q}))'$ for all $j = 1, 2, \ldots$. Therefore, the coefficient at D^{jr+q} in $\mathfrak{L}_j$ vanishes, and $\deg(\mathfrak{L}_j) < q + rj$. On the other hand, it is immediate that $L_{\boldsymbol{S}}(\mathfrak{L}_j) = (rj + q)\zeta \mathfrak{L}_j$. This is in contradiction with the formula $\mathrm{wt}_{\boldsymbol{S}}(\mathfrak{L}_j) = \zeta \deg \mathfrak{L}_j$ unless $\mathfrak{L}_j = 0$, and the result follows. □

Remark. The above proof can be readily extended to include scalings $\boldsymbol{S}$ of more general form and to handle the case when the coefficients of $\mathfrak{R}$ involve nonlocal variables from the universal Abelian covering over $\boldsymbol{u}_\tau = \boldsymbol{Q}$.

Theorem 1 together with propositions 1 and 3 yields the following assertion.

Corollary 3. *Under the assumptions of proposition 3 suppose that $\mathfrak{R}$ is normal, and at least one of the following conditions is satisfied:*

(i) $L_{\boldsymbol{Q}}(\gamma_\alpha) = 0, \alpha = 1, \ldots, p;$

(ii) $\boldsymbol{G}_\alpha, \alpha = 1, \ldots, p$, *are linearly independent over* $\mathbb{T}$ *and* $L_{\boldsymbol{Q}}(\boldsymbol{G}_\alpha) = 0, \alpha = 1, \ldots, p;$

(iii) $\boldsymbol{G}_\alpha, \alpha = 1, \ldots, p$, *are linearly independent over* $\mathbb{T}$ *and* $\mathfrak{R}(\boldsymbol{Q})$ *is local.*

Then $\boldsymbol{Q}_j = \mathfrak{R}^j(\boldsymbol{Q})$ *are local and commute for all* $j = 0, 1, 2, \ldots$.

4. Higher recursion, Hamiltonian and symplectic operators

Consider an operator $\mathfrak{R}$ of the form (5) and another operator of similar form:

$$\tilde{\mathfrak{R}} = \sum_{i=0}^{\tilde{r}} \tilde{a}_i D^i + \sum_{\alpha=1}^{\tilde{p}} \tilde{\boldsymbol{G}}_\alpha \otimes D^{-1} \circ \tilde{\gamma}_\alpha. \tag{6}$$

For a moment we do *not* assume that $\mathfrak{R}$ and $\tilde{\mathfrak{R}}$ act on $\mathcal{V}$, so we do not specify whether the quantities $\boldsymbol{G}_\alpha, \gamma_\alpha, \tilde{\boldsymbol{G}}_\alpha, \tilde{\gamma}_\alpha$ belong to $\mathcal{V}$ or to $\mathcal{V}^*$.

Using the lemma from section 2 of [21] we readily find that

$$(\mathfrak{R} \circ \tilde{\mathfrak{R}})_- = \sum_{\alpha=1}^{\tilde{p}} \mathfrak{R}(\tilde{\boldsymbol{G}}_\alpha) \otimes D^{-1} \circ \tilde{\gamma}_\alpha + \sum_{\alpha=1}^{p} \boldsymbol{G}_\alpha \otimes D^{-1} \circ \tilde{\mathfrak{R}}^\dagger(\gamma_\alpha). \tag{7}$$

Repeatedly using (7) yields the following formulae that hold for integer $n, m \geqslant 1$,

$$(\mathfrak{R}^n)_- = \sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!j!} \left(\sum_{\alpha=1}^{p} \mathfrak{R}^j(\boldsymbol{G}_\alpha) \otimes D^{-1} \circ (\mathfrak{R}^\dagger)^{n-1-j}(\gamma_\alpha) \right), \tag{8}$$

$$((\mathfrak{R}^\dagger)^n)_- = -\sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!j!} \left(\sum_{\alpha=1}^{p} \mathfrak{R}^{\dagger j}(\gamma_\alpha) \otimes D^{-1} \circ \mathfrak{R}^{n-1-j}(\boldsymbol{G}_\alpha) \right), \tag{9}$$

$$(\mathfrak{R}^n \circ \tilde{\mathfrak{R}}^m)_- = \sum_{j=0}^{n-1} \frac{(n-1)!}{(n-1-j)!j!} \left(\sum_{\alpha=1}^{p} \mathfrak{R}^j(\boldsymbol{G}_\alpha) \otimes D^{-1} \circ \tilde{\mathfrak{R}}^{\dagger m} (\mathfrak{R}^\dagger)^{n-1-j}(\gamma_\alpha) \right)$$
$$+ \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-1-j)!j!} \left(\sum_{\alpha=1}^{\tilde{p}} \mathfrak{R}^n \tilde{\mathfrak{R}}^j(\tilde{\boldsymbol{G}}_\alpha) \otimes D^{-1} \circ (\tilde{\mathfrak{R}}^\dagger)^{m-1-j}(\tilde{\gamma}_\alpha) \right). \tag{10}$$

Corollary 2, combined with (7)–(10), immediately yields the following result.

Corollary 4. *Suppose that $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ meets the requirements of corollary 2, and $\mathfrak{P} : \mathcal{V}^* \to \mathcal{V}$, $\mathfrak{S} : \mathcal{V} \to \mathcal{V}^*$, $\mathfrak{N} : \mathcal{V} \to \mathcal{V}$, $\mathfrak{T} : \mathcal{V}^* \to \mathcal{V}^*$ are purely differential operators.*

Then $\mathfrak{R}^k, \mathfrak{R}^{\dagger k}, \mathfrak{P} \circ \mathfrak{R}^{\dagger k}, \mathfrak{S} \circ \mathfrak{R}^k, \mathfrak{N}^q \circ \mathfrak{R}^k$, and $\mathfrak{T}^q \circ \mathfrak{R}^{\dagger k}$ are weakly nonlocal for all $k, q = 0, 1, 2, \ldots$.

If $\mathfrak{B}$ is a scalar differential operator of degree b, then [25] $\dim_{\mathbb{T}}(\mathcal{A} \bigcap \ker \mathfrak{B}) \leqslant b$, and using lemma 2 (see the appendix) we can readily prove the following assertion.

Corollary 5. *Let $s = 1$. Assume that $\mathfrak{R}$ and $\mathfrak{P}$ (resp. $\mathfrak{S}$) meet the requirements of corollary 4, $\deg \mathfrak{P} = b$ (resp. $\deg \mathfrak{S} = b$), and $\mathfrak{R}^{\dagger j}(\gamma_\alpha)$ (resp. $\mathfrak{R}^j(\boldsymbol{G}_\alpha)$) are linearly independent over $\mathbb{T}$ for all $j = 0, \ldots, n-1$ and $\alpha = 1, \ldots, p$.*

Then there exist at most $[b/p]$ local linear combinations of $\mathfrak{P} \circ \mathfrak{R}^{\dagger k}$ (resp. $\mathfrak{S} \circ \mathfrak{R}^k$), $k = 1, \ldots, n$, and any such local linear combination involves only $\mathfrak{P} \circ \mathfrak{R}^{\dagger k}$ (resp. $\mathfrak{S} \circ \mathfrak{R}^k$) with $k \leqslant [b/p]$.

If $\mathfrak{P}$ is a Hamiltonian operator (resp. if $\mathfrak{S}$ is a symplectic operator), the above results, especially corollary 5, enable us to obtain an estimate for the number of local, i.e., purely differential, Hamiltonian (resp. symplectic) operators among the linear combinations of $\mathfrak{P} \circ \mathfrak{R}^{\dagger k}$ (resp. $\mathfrak{S} \circ \mathfrak{R}^k$). Such estimates play an important role, e.g., in the construction of Miura-type transformations [2].

Finally, using propositions 1 and 2 we can readily generalize corollary 4 to the case of weakly nonlocal $\mathfrak{P}, \mathfrak{S}, \mathfrak{T}, \mathfrak{N}$ as follows:

Theorem 2. *Suppose that $\mathfrak{R} : \mathcal{V} \to \mathcal{V}$ of the form (5) meets the requirements of corollary 2, and $\boldsymbol{K}_\beta, \boldsymbol{H}_\beta \in \mathcal{V}$ and $\boldsymbol{\eta}_\beta, \boldsymbol{\zeta}_\beta \in \mathcal{V}^*$ are such that $L_{\boldsymbol{K}_\beta}(\mathfrak{R}) = 0, L_{\boldsymbol{H}_\beta}(\mathfrak{R}) = 0, \boldsymbol{\eta}'_\beta = \boldsymbol{\eta}'^{\dagger}_\beta, \boldsymbol{\zeta}'_\beta = \boldsymbol{\zeta}'^{\dagger}_\beta, (\mathfrak{R}^\dagger(\boldsymbol{\eta}_\beta))' = (\mathfrak{R}^\dagger(\boldsymbol{\eta}_\beta))'^{\dagger}, (\mathfrak{R}^\dagger(\boldsymbol{\zeta}_\beta))' = (\mathfrak{R}^\dagger(\boldsymbol{\zeta}_\beta))'^{\dagger}, L_{\boldsymbol{K}_\beta}(\gamma_\alpha) = 0, L_{\boldsymbol{H}_\beta}(\gamma_\alpha) = 0, L_{\boldsymbol{G}_\alpha}(\boldsymbol{\eta}_\beta) = 0$ and $L_{\boldsymbol{G}_\alpha}(\boldsymbol{\zeta}_\beta) = 0$ for all $\alpha = 1, \ldots, p$ and $\beta = 1, \ldots, m$. Further assume that $\mathfrak{P} : \mathcal{V}^* \to \mathcal{V}$, $\mathfrak{S} : \mathcal{V} \to \mathcal{V}^*$, $\mathfrak{T} : \mathcal{V}^* \to \mathcal{V}^*$ and $\mathfrak{N} : \mathcal{V} \to \mathcal{V}$ are weakly nonlocal and we have $\mathfrak{P}_- = \sum_{\beta=1}^m \boldsymbol{K}_\beta \otimes D^{-1} \circ \boldsymbol{H}_\beta$, $\mathfrak{S}_- = \sum_{\beta=1}^m \boldsymbol{\zeta}_\beta \otimes D^{-1} \circ \boldsymbol{\eta}_\beta$, $\mathfrak{T}_- = \sum_{\beta=1}^m \boldsymbol{\zeta}_\beta \otimes D^{-1} \circ \boldsymbol{K}_\beta$ and $\mathfrak{N}_- = \sum_{\beta=1}^m \boldsymbol{H}_\beta \otimes D^{-1} \circ \boldsymbol{\eta}_\beta$.*

Then $\mathfrak{P} \circ \mathfrak{R}^{\dagger k}, \mathfrak{T} \circ \mathfrak{R}^{\dagger k}, \mathfrak{S} \circ \mathfrak{R}^k$, and $\mathfrak{N} \circ \mathfrak{R}^k$ are weakly nonlocal for all $k = 0, 1, 2, \ldots$.

Note that if $\mathfrak{P}$ is a Hamiltonian operator and $\mathfrak{S}$ is a symplectic operator, then they are skew-symmetric ($\mathfrak{P}^\dagger = -\mathfrak{P}$ and $\mathfrak{S}^\dagger = -\mathfrak{S}$), and we can set without loss of generality $\boldsymbol{H}_\beta = \epsilon_\beta \boldsymbol{K}_\beta$ and $\boldsymbol{\zeta}_\beta = \tilde{\epsilon}_\beta \boldsymbol{\eta}_\beta$, where ϵ_β and $\tilde{\epsilon}_\beta$ are constants taking one of three values, -1, 0 or $+1$, see e.g. [23]. The conditions of theorem 2 for $\boldsymbol{\zeta}_\beta$ and $\boldsymbol{H}_\beta$ are then automatically satisfied. Moreover, if $\mathfrak{R}$ is a recursion operator, $\mathfrak{P}$ is a Hamiltonian operator and $\mathfrak{S}$ is a symplectic operator for an integrable system in (1+1) dimensions, then theorem 2 proves, under some natural assumptions that are satisfied for virtually all known examples, the Maltsev–Novikov conjecture which states [10] that higher recursion operators $\mathfrak{R}^k$, higher Hamiltonian operators $\mathfrak{P} \circ \mathfrak{R}^{\dagger k}$ and higher symplectic operators $\mathfrak{S} \circ \mathfrak{R}^k$ are weakly nonlocal for all $k = 0, 1, 2, \ldots$.

5. Examples

Consider a hereditary recursion operator (see, e.g., the discussion on page 122 of [28] and references therein)

$$\mathfrak{R} = D^2 + 2au_1^2 + \tfrac{4}{3}bu_1 + c - \tfrac{2}{3}(3au_1 + b)D^{-1} \circ u_2$$

for the generalized potential modified Korteweg–de Vries equation

$$u_t = u_3 + au_1^3 + bu_1^2 + cu_1,$$

where a, b, c are arbitrary constants. This operator meets the requirements of theorem 1 for $\boldsymbol{Q} = u_1$, so all $\boldsymbol{Q}_j = \mathfrak{R}^j(\boldsymbol{Q})$, $j = 1, 2, \ldots$, are local.

The equation in question has infinitely many Hamiltonian operators $\mathfrak{P} = D$ and $\mathfrak{P}_j = \mathfrak{P} \circ \mathfrak{R}^{\dagger j}$, $j \in \mathbb{N}$ (in particular, we have $\mathfrak{P}_1 = D^3 + \left(2au_1^2 + \tfrac{4}{3}bu_1 + c\right)D - \tfrac{2}{3}(3au_1 + b)u_2 + \tfrac{2}{3}(3au_1 + b)D^{-1} \circ u_1$). By corollary 4 all $\mathfrak{P}_j$, $j = 1, 2, \ldots$, are weakly nonlocal, and by corollary 5 $\mathfrak{P}$ is the only *local* Hamiltonian operator among $\mathfrak{R}^j \circ \mathfrak{P}$ for $j = 0, 1, 2 \ldots$.

For another example, consider a linear combination of the Harry Dym equation and the time-independent parts of its scaling symmetries, cf e.g. [2, 18, 17]:

$$u_t = u^3u_3 + axu_1 + bu, \tag{11}$$

where a and b are arbitrary constants, and a hereditary recursion operator for (11)

$$\begin{aligned}\mathfrak{R} &= \exp(-3(a+b)t)u^3D^3 \circ u \circ D^{-1} \circ \exp((a+b)t)/u^2 \\ &= \exp(-2(a+b)t)(u^2D^2 - uu_1D + uu_2) + \exp(-3(a+b)t)u^3u_3D^{-1} \circ \exp((a+b)t)/u^2.\end{aligned}$$

Again, the requirements of theorem 1 are met for $\boldsymbol{Q} = \exp(-3(a+b)t)u^3u_3$, so all $\boldsymbol{Q}_j = \mathfrak{R}^j(\boldsymbol{Q})$, $j = 1, 2, \ldots$, are local.

Note that in both of these examples there is no scaling symmetry of the form used in [11], and hence the locality of corresponding hierarchies cannot be established by direct application of the results from [11].

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Appendix

Here we prove the following lemma kindly communicated to the author by V V Sokolov.

Lemma 2. *Consider* $\mathfrak{H} = \sum_{\alpha=1}^{m} \vec{f}_\alpha \otimes D^{-1} \circ \vec{g}_\alpha$, *where* $\vec{f}_\alpha, \vec{g}_\alpha \in \mathcal{A}^q$, *and* $\vec{f}_\alpha$ *are linearly independent over the field* $\mathbb{T}$ *of locally analytic functions of* t.

Then $\mathfrak{H} = 0$ *if and only if* $\vec{g}_\alpha = 0$ *for all* $\alpha = 1, \ldots, m$.

Proof. Clearly, $\mathfrak{H} = 0$ if and only if $\mathfrak{H}^\dagger = 0$. Using (2) we find that

$$\mathfrak{H}^\dagger = -\sum_{j=0}^{\infty}\sum_{\alpha=1}^{m}(-1)^j \vec{g}_\alpha \otimes D^j(\vec{f}_\alpha)D^{-1-j}.$$

Equating to zero the coefficients at powers of D in $\mathfrak{H}^\dagger = 0$, we obtain the following system of linear *algebraic* equations for $\vec{g}_\alpha$:

$$\sum_{\alpha=1}^{m} g_\alpha^k D^j\big(f_\alpha^d\big) = 0, \qquad d, k = 1, \ldots, q; \quad j = 0, 1, 2, \ldots. \tag{A.1}$$

We want to prove that the linear independence of $\vec{f}_\alpha$ over $\mathbb{T}$ implies that $g_\alpha^k = 0$ for all α and k. To this end let us first fix k and consider (A.1) as a system of linear equations for the components g_α^k of $\vec{g}_\alpha$.

Clearly, if the rank ρ of the matrix of this system equals m, then $g_\alpha^k = 0$, so we can prove our claim by proving that if $\rho < m$, then $\vec{f}_\alpha$ are *linearly dependent* over $\mathbb{T}$. Indeed, if $\rho < m$, then the columns of our matrix are linearly dependent over $\mathcal{A}$. On the other hand, ρ of them must be linearly independent over $\mathcal{A}$. Assume without loss of generality that these are just the first ρ columns. The rest can be expressed via them, that is, there exist $h_\beta^\alpha \in \mathcal{A}$ such that

$$D^j(\vec{f}_\beta) = \sum_{\alpha=1}^{\rho} h_\beta^\alpha D^j(\vec{f}_\alpha), \qquad \beta = \rho+1, \ldots, m, \quad j = 0, 1, 2, \ldots. \tag{A.2}$$

As h_β^α are independent of j, the consistency of the above equations and the linear independence of first ρ columns over $\mathcal{A}$ imply that $D\big(h_\beta^\alpha\big) = 0$, hence $h_\beta^\alpha = h_\beta^\alpha(t)$, and (A.2) for $j = 0$ implies the linear dependence of $\vec{f}_\alpha$ over $\mathbb{T}$, which contradicts our initial assumptions. Thus, if $\vec{f}_\alpha$, $\alpha = 1, \ldots, m$, are linearly independent over $\mathbb{T}$, then the matrices in question are of rank m for all k, and hence $\vec{g}_\alpha = 0$ for all $\alpha = 1, \ldots, m$. □

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